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Conformal invariance and the critical behaviour of a quantum spin Hamiltonian with three-spin coupling

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Abstract. A quantum spin model with three-spin coupling is studied by finite-lattice methods. The critical indices ν and α are obtained by standard finite-size scaling analysis. Conformal invariance is used to estimate the exponents ν and η, the central charge or conformal anomaly c and the anomalous dimension of a spin-$\frac{1}{4}$ parafermion. We conclude that the model belongs to the same universality class as the four-state Potts and Baxter-Wu models, although the convergence of the various finite lattice estimators is extremely slow.

1. Introduction

Our knowledge of the critical behaviour of systems involving multispin interactions is rather limited compared with that of systems with two-spin interactions. Most of our understanding of the effects of multispin couplings on critical behaviour comes from two exactly soluble models: the eight-vertex model (Baxter 1972), which in magnetic language involves both two- and four-spin interactions, and the Baxter-Wu model (Baxter and Wu 1974, Baxter 1974), in which the only interactions are three-spin interactions around the elementary faces of a triangular lattice. Both models exhibit distinctive critical behaviour and reveal a diversity of critical behaviour that may be produced by multispin interactions.

More recently (Turban 1982a, Penson *et al* 1982, Debierre and Turban 1983) have introduced a two-dimensional Ising model with three-spin interactions in one direction (which we will call the 'space' direction) and simple two-spin interactions in the other direction ('time' direction). The Hamiltonian of the model is

$$H = -\sum_{(i,j)} \{J_s\sigma_{i,j}\sigma_{i+1,j}\sigma_{i+2,j} + J_t\sigma_{i,j}\sigma_{i,j+1}\} \tag{1.1}$$

where the summation is over all sites of a (square) lattice, $J_s > 0$ and $J_t > 0$ are coupling constants and $\sigma_{i,j} = \pm 1$ are classical Ising variables. This Hamiltonian is self-dual (Turban 1982a, Debierre and Turban 1983), reflecting a general property of systems with two interactions per lattice site (Alcaraz 1982).

A time continuous quantum Hamiltonian of (1.1):

$$H = -\sum_i \sigma_i^z\sigma_{i+1}^z\sigma_{i+2}^z - \lambda \sum_i \sigma_i^x \tag{1.2}$$

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may be obtained by standard procedures (Fradkin and Susskind 1978), where σ_i^x, σ_i^z are the spin-$\frac{1}{2}$ Pauli matrices and λ is a coupling constant. We shall refer to this Hamiltonian as the 'three-spin transverse Ising model'. The Hamiltonian is also self-dual, implying, on the assumption of a single transition, that the critical coupling is $\lambda_c = 1$ (Penson *et al* 1982, Turban 1982b).

The ground states of both (1.1) and (1.2) are four-fold degenerate; the possible states consisting of repetitions of the patterns $+++$, $+--$, $-+-$, $--+$, respectively. The relevant symmetry of the model is a semi-global $Z(2)\otimes Z(2)$ symmetry, which can be seen by partitioning the lattice into three sublattices such that the three-spin interaction involves a site from each sublattice. The Hamiltonians (1.1) and (1.2) are then symmetric under the reversal of all spins on any two sublattices. This symmetry is identical to that found in the Baxter–Wu model.

This degeneracy and symmetry imply that the appropriate order parameter to describe the ordered phase has four components with an effective Landau free energy functional identical to the common functional describing criticality in the Baxter–Wu and the four-state Potts models (see, e.g., Barber 1980). These symmetry considerations lead to the conjecture (Debierre and Turban 1983, Maritan *et al* 1984) that both (1.1) and (1.2) belong to the same universality class as the Baxter–Wu and four-state Potts models with exponents (Baxter and Wu 1974, den Nijs 1979)

$$\alpha = \tfrac{2}{3} \qquad \nu = \tfrac{2}{3} \qquad \eta = \tfrac{1}{4}. \tag{1.3}$$

Direct tests of this conclusion, largely by finite-lattice methods, have been inconclusive to date. In the transfer matrix formalism, Debierre and Turban (1983) performed calculations on $m \times \infty$ strips with periodic boundary conditions for $m = 3, 6, 9$. While no definite conclusion on the value of exponents could be drawn, their results were not inconsistent with four-state Potts behaviour. In the Hamiltonian formalism (1.2) two different results have been reported. Penson *et al* (1982), on the basis of data from lattices of up to 15 sites with periodic boundary conditions, estimated $\nu \approx 0.78$ but reserved any final conclusion on the universality class. On the other hand, Iglói *et al* (1983) estimated $\nu = 0.77$ from lattices of up to nine sites with free edge boundary conditions and claimed that the model was in a different universality class from the four-state Potts model.

In order to verify whether or not the exponent values (1.3) can be confidently excluded, we report in this paper a careful finite-lattice study of the quantum Hamiltonian (1.2). Our analysis has two phases. We first extend the Penson *et al* (1982) calculation to lattice size $m = 18$. Secondly, we exploit recent predictions of conformal invariance (Cardy 1986b, von Gehlen *et al* 1986) concerning the eigenvalue spectrum of a transfer matrix (or quantum Hamiltonian) of a two-dimensional system in a strip of finite width.

The paper is arranged as follows. In the next section we describe our conventional finite-size scaling analysis. The exploitation of the predictions of conformal invariance is carried out in § 3. The paper closes with an overall summary and discussion in § 4.

2. Finite-size scaling

The calculation of low-lying eigenenergies of a Hamiltonian such as (1.1) on a lattice of m sites is now standard (see, e.g., Hamer and Barber 1981a). We write

$$H = H_0 + V \tag{2.1}$$

with

$$H_0 = -\lambda \sum \sigma_j^x \qquad V = -\sum \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^z \tag{2.2}$$

and use the eigenstates of H_0 as a basis. In this basis, the Hilbert space can be decomposed into disjoint sectors labelled by the eigenvalues of three parity operators

$$\mathscr{P}_1 = \prod_{i=1}^{m/3} \sigma_{3i}^x \sigma_{3i+1}^x \qquad \mathscr{P}_2 = \prod_{i=1}^{m/3} \sigma_{3i}^x \sigma_{3i+2}^x \qquad \mathscr{P}_3 = \mathscr{P}_1 \cdot \mathscr{P}_2 \tag{2.3}$$

which independently commute with (2.1) for all λ. With periodic boundary conditions applied these sectors can be further block diagonalised according to the eigenvalues of the translation operator (linear momentum operator).

We initially focus attention on the two lowest eigenvalues, $E_0(\lambda; m)$ and $E_1(\lambda; m)$, of (2.1). The former is the lowest energy state in the zero momentum sector with $\mathscr{P}_1 = \mathscr{P}_2 = 1$, while E_1 is three-fold degenerate; the eigenstates being the lowest lying zero momentum states in the sectors $\mathscr{P}_1 = \mathscr{P}_2 = -1$, and $\mathscr{P}_1 = -\mathscr{P}_2 = 1$, $\mathscr{P}_1 = \mathscr{P}_2 = -1$ respectively. To evaluate E_0 and E_1 we used the Lanczos method (Hamer and Barber 1981a, Roomany *et al* 1980) starting respectively from the states $|\phi_0\rangle$ and $(\Sigma\, \sigma_m^z/\sqrt{M})|\phi_0\rangle$ where $|\phi_0\rangle$ is the ground state of H_0.

From E_0 and E_1 (and their derivatives with respect to λ) we obtain the mass gap

$$\Lambda_m(\lambda) = E_1(\lambda; m) - E_0(\lambda; m) \tag{2.4}$$

the Callan–Symanzik β function (Hamer *et al* 1979)

$$\beta_m(\lambda) = -\Lambda_m(\lambda)[\Lambda_m(\lambda) - 2\lambda\, \partial\Lambda_m/\partial\lambda]^{-1} \tag{2.5}$$

and the analogue of the specific heat per site

$$c_m(\lambda) = -(\lambda^2/m)\partial^2 E_0(\lambda; m)/\partial\lambda^2. \tag{2.6}$$

Values of these quantities at the self-dual (and expected critical point) $\lambda = 1$ are listed in table 1 for lattices of up to 18 sites. Note that m must be an integral multiple of three to fit the allowed ground states. These data form the basis of our finite-size scaling analysis.

Unfortunately, even data from 18 sites appears to be insufficient to allow an accurate estimate of critical exponents. Certainly, sophisticated extrapolation methods (see e.g., Hamer and Barber 1981b) cannot be reliably applied. The best procedure we have

Table 1. Finite-lattice data for the three-spin transverse Ising model (1.2) as a function of lattice size m. Listed are values at $\lambda = \lambda_c = 1$ for the ground-state energy per site E_0/m, the specific heat c_m, the mass gap Λ_m and the β function β_m.

m	$-E_0/m$	c_m	Λ_m	β_m
3	1.414 214	0.353 553	1.080 363	0.427 051
6	1.247 219	0.846 327	0.486 729	0.158 677
9	1.219 590	1.225 855	0.315 628	0.090 683
12	1.210 202	1.561 670	0.233 680	0.061 117
15	1.205 910	1.871 693	0.185 522	0.045 013
18	1.203 597	2.163 100	0.153 808	0.035 062

found is to use a simple ratio analysis (Gaunt and Guttmann 1974) of the sequences for β_m and c_m. On the basis of finite-size scaling, we expect (Barber 1983)

$$\beta_m(\lambda_c) \sim m^{-1/\nu} \qquad m \to \infty \tag{2.7}$$

$$c_m(\lambda_c) \sim m^{\alpha/\nu} \qquad m \to \infty. \tag{2.8}$$

Hence for large m, the ratios

$$\beta_m(\lambda_c)/\beta_{m-3}(\lambda_c) = 1 - \frac{3/\nu}{m} + o\left(\frac{1}{m}\right) \tag{2.9}$$

and

$$c_m(\lambda_c)/c_{m-3}(\lambda_c) = 1 + \frac{3\alpha/\nu}{m} + o\left(\frac{1}{m}\right) \tag{2.10}$$

allow in principle estimates of $1/\nu$ and α/ν. These ratio plots are shown in figure 1. Considerable curvature in the plots is evident. The plots can be straightened by 'ε-shifting', i.e. by plotting the ratios against $1/(m+\varepsilon)$. In this way, we obtain

$$\nu \approx 0.73 \qquad \alpha/\nu \approx 0.72 \tag{2.11}$$

the estimate for ν agreeing with that of Penson *et al* (1982). However, the extreme curvature evident in the unshifted plots and the size of the shift needed to remove it makes us very hesitant in concluding that the available data even for $m = 18$ can be regarded as having reached the asymptotic regime. We shall return to this point in our concluding discussion in § 4.

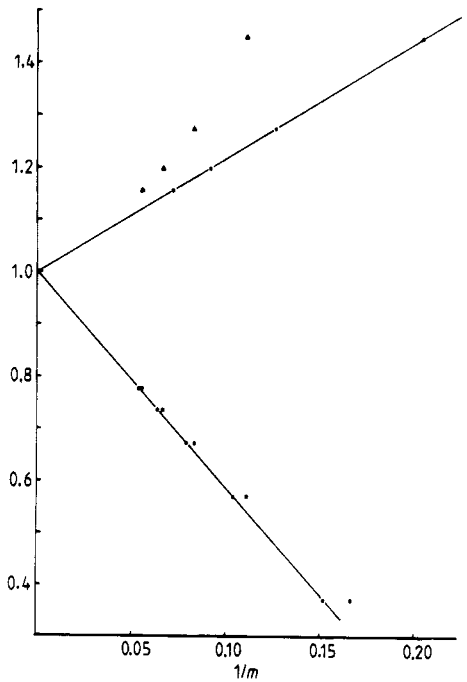

Figure 1. Ratio plots of β_m/β_{m-3} (■, lower part of figure) and c_m/c_{m-3} (▲, upper part of figure) against $1/m$. The straight lines were obtained by plotting against $1/(m+0.6)$ for β_m/β_{m-3} and $1/(m-4.1)$ for c_m/c_{m-3}.

3. Conformal invariance

Statistical mechanical systems at criticality are believed to be conformally invariant: an assumption that in two dimensions has many significant implications (for a recent review see Cardy 1986b). In particular, Cardy (1984, 1986a) has derived a set of remarkable relations between the eigenvalue spectrum of the transfer matrix in a strip of finite width and the anomalous dimensions of the operator algebra describing the critical behaviour of the infinite system. These results can be transcribed (von Gehlen *et al* 1986) to the quantum Hamiltonian formalism in which we are interested.

The pertinent results for our purposes are as follows. Decompose the state space of (2.1) as before into the 'ground-state' sector ($\mathcal{P}_1 = \mathcal{P}_2 = 1$) and the degenerate 'excited-state' sectors ($\mathcal{P}_1 = \mathcal{P}_2 = 1$, $\mathcal{P}_1 = -\mathcal{P}_2 = -1$, $\mathcal{P}_1 = \mathcal{P}_2 = -1$). Let $E_{0,k}$, $E_{1,k}$, $k = 0, 1, \ldots$, be the successive energy levels in these two sectors. Then conformal invariance implies that at the bulk critical coupling ($\lambda = \lambda_c = 1$) and as $m \to \infty$

$$E_{1,0} - E_{0,0} = \Lambda_m = \frac{2\pi x_\sigma \zeta}{m} + \mathrm{o}\left(\frac{1}{m}\right) \tag{3.1}$$

$$E_{1,1} - E_{1,0} = \frac{2\pi\zeta}{m} + \mathrm{o}\left(\frac{1}{m}\right) \tag{3.2}$$

$$E_{0,1} - E_{0,0} = \frac{2\pi x_\varepsilon \zeta}{m} + \mathrm{o}\left(\frac{1}{m}\right) \tag{3.3}$$

where $x_\sigma = \eta/2$ and $x_\varepsilon = d - 1/\nu$ are the anomalous dimensions of the order and energy operators respectively. The constant ζ does not appear in the transfer matrix formalism but enters the Hamiltonian relations since the Hamiltonian may in principle be multiplied by an arbitrary constant (Alcaraz and Drugowich de Felício 1984, Penson and Kolb 1984, Alcaraz *et al* 1985, von Gehlen *et al* 1986).

In addition to these relations on gaps at criticality, conformal invariance predicts (Blöte *et al* 1986) that the ground-state energy per site should approach its bulk limit, e_0, as

$$E_0/m = e_0 - \tfrac{1}{6}\pi c\zeta/m^2 + \mathrm{o}(1/m^2) \tag{3.4}$$

where c is the central charge or conformal anomaly of the appropriate conformal class of the transition in the bulk system. Friedan *et al* (1984) have argued that for unitary theories with $c < 1$, c is restricted to the countable set of values

$$c = 1 - 6/n(n+1) \qquad n = 3, 4, 5, \ldots \tag{3.5}$$

including $c = 1$. They showed that unitarity places no constraint for $c \geqslant 1$.

Extensive numerical tests on various models have confirmed these relations as powerful probes of bulk critical behaviour given finite-lattice data. However, before applying these ideas to the three-spin transverse Ising model two remarks are in order. The first concerns the validity of the assumption that the two-dimensional model (1.1) itself is conformally invariant at criticality let alone the quantum Hamiltonian version (1.2). The interactions in (1.1) are clearly not isotropic. However, by an appropriate choice of J_s and J_t, say J_s^* and J_t^*, the *correlation function* can be made asymptotically rotationally invariant. This model should then satisfy the basic assumptions—short-ranged interactions, scale invariance, rotational and translational invariance—that ensure conformal invariance (see Cardy 1986b). For $J_s \neq J_s^*$ and $J_t \neq J_t^*$, the decay of the correlation function may no longer be rotationally invariant. However, such a decay can be recovered by *anisotropically* scaling the x and y directions differently. Since this is the same physical concept that underlies the quantum Hamiltonian limit (Fradkin and Susskind 1978) the quantum Hamiltonian itself should also be conformally invariant. The only problem is that the required degree of anisotropic scaling is unknown *a priori* giving rise to the (unknown) constant ζ that appears in (3.1)-(3.4).

The second remark that should be made concerns the *numerical* calculation of the eigenvalues entering (3.1)-(3.3). Here the Lanczos scheme is particularly useful (see, e.g., Cullum and Willoughby 1981); the required eigenvalues following from the successive eigenvalues of the tridiagonal matrix generated by the Lanczos algorithm. We had no problem in recognising 'spurious' unphysical eigenvalues that arise from loss of orthogonalisation due to round-off error (Cullum and Willoughby 1981, Alcaraz and Drugowich de Felício 1985).

Table 2 lists values of $m\Delta E$ for the four energy levels entering (3.1)-(3.3) along with estimates of x_σ and x_ε that follow by dividing (3.1) and (3.3) by (3.2) respectively. Unfortunately, the brevity of the lattice data again limits the applicability of acceleration techniques. The sequence of estimates of x_σ can be accelerated by standard accelerators (see, e.g., Smith and Ford 1979, Barber and Hamer 1982) yielding $x_\sigma \approx 0.13$ consistent with $\eta = \frac{1}{4}$. On the other hand, the sequence of estimates of x_ε is much less well behaved and the results of different acceleration techniques differ considerably. The best that one can definitely conclude from this approach is that $x_\varepsilon < 0.65$ implying $\nu < 0.74$ and that a lower value of x_ε and hence ν is extremely likely.

We have also attempted to estimate the conformal anomaly c from (3.4). This is complicated by the necessity of estimating the (unknown) infinite lattice limit e_0. Estimates of e_0 and $b = -\pi c\zeta/6$ obtained by fitting E_0 to the form

$$E_0 = e_0 m + b/m \tag{3.6}$$

using data from two different lattices are given in table 3. These results suggest $b \sim -1.7$ which taking $\zeta \sim 20/2\pi$ implies $c \sim 1.02$.

Table 2. Mass gap amplitudes and anomalous dimension estimates for three-spin transverse Ising model (1.2) as a function of lattice size.

m	$m(E_{0,1}-E_{0,0})$	$m(E_{1,1}-E_{1,0})$	$m(E_{1,0}-E_{0,0})$	x_ε	x_σ
6	15.5060	14.5206	2.9204	1.0679	0.2011
9	14.8263	18.0770	2.8407	0.8202	0.1571
12	14.4298	19.1208	2.8042	0.7547	0.1467
15	14.1618	19.5640	2.7828	0.7239	0.1422
18	13.9644	19.7927	2.7685	0.7055	0.1399

Table 3. Estimates of e_0 and $b = -\pi c\zeta/6$ (see text) obtained from two-point fits to the ground-state energy.

m_1	m_2	e_0	b	c	c/η
6	9	−1.197 49	−1.790 3	1.19	3.781
9	12	−1.198 13	−1.738 2	1.09	3.719
12	15	−1.198 28	−1.716 65	1.05	3.701
15	18	−1.198 33	−1.705 45	1.03	3.696

Alternatively, we can eliminate the constant ζ by dividing the estimate of b obtained from lattice pair (m_1, m_2) by the amplitudes $m_2(E_{1,1}-E_{1,0})$ or $m_2(E_{1,0}-E_{0,0})$ from table 2 leading to sequences that should tend to $c/12$ and $c/6\eta$ respectively. The resulting estimates of c and c/η are shown in the right-hand two columns of table 3. While the latter sequences are apparently monotonic decreasing *away* from $c=1$, we believe that this is a finite-lattice effect and that $c=1$ cannot be confidently excluded.

Support for this view comes from similar estimations (see table 4) of the ratio c/η for the quantum Hamiltonian four-state Potts model and the Baxter-Wu model (transfer matrix formulation) using the published finite-lattice data of Hamer (1981) and Barber (1985) respectively. The four-state Potts estimates of c/η are remarkably comparable to those obtained for the three-spin transverse Ising model and likewise are decreasing monotonically *away* from the expected value of 4. On the other hand, the Baxter-Wu data appear to be converging to the expected limit.

Finally, following von Gehlen *et al* (1986), we have looked for gaps in the spectrum of H with $\lambda=\lambda_c=1$ associated with the anomalous dimensions of parafermion operators (Fradkin and Kadanoff 1980) by applying different boundary conditions. Specifically, with antiperiodic boundary conditions and a non-cyclic initial Lanczos vector we found a new eigenvalue $E_1{}^A$ that did not appear in the spectrum for periodic boundary conditions. The difference between this eigenvalue and the ground state $E_{0,0}$ with periodic boundary conditions is tabulated in table 5 along with the anomalous

Table 4. Estimates of the ratio c/η for (a) the quantum Hamiltonian four-state Potts model and (b) the Baxter-Wu model.

(a)

m_1	m_2	e_0	b	$m_2\Lambda(m_2)$	c/η
3	4	−3.5406	−1.7679	2.8307	3.7473
4	5	−3.5435	−1.7216	2.7955	3.6951
5	6	−3.5444	−1.6988	2.7729	3.6759
6	7	−3.5448	−1.6857	2.7569	3.6686
7	8	−3.5449	−1.6773	2.7449	3.6665

(b)

m_1	m_2	e_0	b	$m_2\Lambda(m_2)$	c/η
3	6	0.015 17	−0.4256	0.6760	3.7775
6	9	0.014 56	−0.4473	0.6788	3.9537
9	12	0.014 52	−0.4509	0.6795	3.9819

Table 5. Estimates of parafermion dimension.

m	$m(E_1{}^A-E_{0,0})$	x_{pf}
6	10.9884	0.7567
9	11.4586	0.6339
12	11.6379	0.6087
15	11.7359	0.5998

dimension x_{pf} that follows by dividing this gap by $m(E_{1,1}-E_{1,0})$ from table 2. Extrapolation of this sequence suggests $x_{pf} \simeq 0.53$–0.56, consistent with the dimension ($\frac{17}{32}=0.531\,125$) of the spin-$\frac{1}{4}$ parafermion operator of the four-state Potts model (von Gehlen *et al* 1986, Nienhuis and Knops 1985).

4. Conclusion and discussion

In the preceding two sections we have presented a detailed analysis of finite-lattice data for the Hamiltonian three-spin transverse Ising (1.2) on lattices of up to $m = 18$ sites. A conventional finite-size analysis (§ 2) yielded the estimates

$$\nu = 0.73 \qquad \alpha \approx 0.52. \tag{4.1}$$

These values are significantly different from the values ($\nu = \alpha = \frac{2}{3}$) expected if this model is in the same universality as the four-state Potts and Baxter–Wu models. However, they are remarkably close to the estimates of

$$\nu = 0.71 \pm 0.02 \qquad \alpha = 0.53 \pm 0.02 \tag{4.2}$$

obtained by Hamer (1981) in a similar finite-size scaling analysis of the four-state Potts model also in a quantum Hamiltonian formulation. For the four-state Potts model, the slow convergence of finite-lattice estimators is attributed to the presence of anomalous logarithmic corrections due to the existence of a marginal field. In contrast, finite-lattice estimators for the Baxter–Wu model (for which the marginal field vanishes) are rapidly convergent (Barber 1985). This view has recently received some direct support by Spronken *et al* (1986) who analysed a model (the staggered Heisenberg chain) which has similar corrections and is also in the four-Potts class. Ignoring the logarithmic corrections and analysing their finite-lattice data in a conventional way gave $\nu \approx 0.71$ but estimators specifically constructed to allow for the logarithmic corrections gave lower values ($\nu \approx 0.67$), in accord with the true result. Unfortunately, to be usefully applied their methods of analysis require more data than we have.

In the preceding section we analysed the spectrum of the three-spin transverse Ising model in more detail drawing on recent predictions of conformal invariance. This analysis yielded the estimates

$$\eta \approx 0.26 \qquad \text{and} \qquad \nu \leqslant 0.74 \tag{4.3}$$

the estimate for η in excellent agreement with (1.3). In addition, we were able to detect a parafermion operator with a dimension (~ 0.53) in close agreement with the value of $\frac{17}{32}$ predicted for the four-state Potts model.

Conformal invariance also allows a direct estimate from finite-lattice data of the conformal anomaly c and hence of the relevant universality class of the model. Unfortunately, the extraction of c from finite-lattice data is delicate, but our results are consistent with $c = 1$.

Since $c = 1$ and $\eta = \frac{1}{4}$ are also the values that pertain to the eight-vertex model or Ashkin–Teller model in which ν varies continuously with coupling, it is conceivable that the three-spin transverse Ising model corresponds to some point on the non-universal line of these models. However, this seems extremely unlikely due to the following argument concerning the structure of the spectrum of (1.2).

Since the Hamiltonian (1.2) is translationally invariant and commutes for all λ with the parity operators $\mathcal{P}_i$ (2.3), the spectra in the three sectors ($\mathcal{P}_1 = \mathcal{P}_2 = -1$,

$\mathscr{P}_1 = -\mathscr{P}_2 = -1$, $\mathscr{P}_1 = -\mathscr{P}_2 = +1$) should be identical. Conformal invariance (Cardy 1986a, b) implies then that we should have three primary operators in the infinite system with equal dimension and spin.

To understand the implications of this degeneracy further, it is convenient to recall the well studied spectrum of the general $Z(2)\otimes Z(2)$ model (Kohmoto *et al* 1981, Alcaraz and Drugowich de Felício 1984) with periodic boundary conditions:

$$-H = \sum_i \{[J_1(\sigma_i^z\sigma_{i+1}^z + \tau_i^z\tau_{i+1}^z) + J_4\sigma_i^z\sigma_{i+1}^z\tau_i^z\tau_{i+1}^z] + [J_1(\sigma_i^x + \tau_i^x) + J_4\sigma_i^x\tau_i^x]\} \tag{4.4}$$

where σ^x, τ^x, σ^z, τ^z are Pauli matrices. The counterparts of the parity operators (2.3) are the operators

$$\theta_1 = \prod_i \sigma_i^x \qquad \theta_2 = \prod_i \tau_i^x \qquad \theta_3 = \theta_1\theta_2 \tag{4.5}$$

which commute with (4.4). Hence in a basis that diagonalises, σ^x and τ^x, the Hilbert space can be separated into sectors according to the eigenvalues of the operators θ_1 and θ_2. The ground state is non-degenerate and belongs to the ground state of the sector in which $(\theta_1, \theta_2) = (+, +)$.

The spectrum of the excited sectors $(+, -)$, $(-, +)$ are equal, due to the $\sigma \leftrightarrow \tau$ symmetry of (4.4). This implies that the anomalous dimension χ_m associated with the σ^z and τ^z operators should be equal; or $\langle\sigma^z(i)\sigma^z(i+n)\rangle = \langle\tau^z(i)\tau^z(i+n)\rangle$. On the other hand, the gap between the lowest eigenvalue of the excited sector $(--)$ and the ground state is related to the dimension χ_p of the polarisation operator $\sigma\tau$, which governs the correlation function $\langle\sigma^z(i)\tau^z(i)\sigma^z(i+n)\tau^z(i+n)\rangle$. Since, for $J_4/J_1 \neq 1$, $x_m \neq x_p$, the sector $(--)$ is different from the other excited ones. However, when $J_4/J_1 = 1$ the Hamiltonian (4.4) becomes the quantum version of the four-state Potts model. In this case, an additional symmetry $(\sigma \leftrightarrow \sigma, \sigma\tau \leftrightarrow \rho)$, makes all three excited sectors degenerate, with the result, as a consequence of conformal invariance, that $\chi_\varepsilon = \chi_m$.

In the three-spin transverse Ising model (1.2), we have exactly the same picture: the three lowest excited states are those governing the two-point correlations in a given sublattice $\langle\sigma^z(3i+k)\sigma^z(3i+k+n)\rangle$, $k = 0, 1, 2$, and the magnetic exponents associated with these correlations should be equal, exactly as in the four-state Potts model. We feel that this is strong evidence that the three-spin transverse Ising model does indeed belong to the same universality class as the four-state Potts model and the Baxter–Wu model. It also raises the interesting possibility of embedding (1.2) in a generalised model whose phase diagram has the same topology as the Ashkin–Teller Hamiltonian (4.4). We develop this idea in a separate report (Alcaraz and Barber 1986).

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